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VARIATIONAL ESTIMATES FOR DISPERSION AND ATTENUATION OF WAVES IN RANDOM COMPOSITES—I

GENERAL THEORY

D. R. S. TALBOT and J. R. WILLIS
School of Mathematics, Bath University, Bath BA2 7AY, England

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Abstract—A recently-developed variational formulation is applied to the study of plane waves in randomly inhomogeneous media. Explicit results (involving definite integrals) are obtained for long waves, propagating in an n-phase medium, without limitation to isotropy of either phases or correlation functions. The estimated overall moduli, and also perturbations to the dispersion relation, which give both dispersion and attenuation of the mean wave, are sensitive to the choice of "comparison medium" which is used in the formulation: it can be chosen, in particular, so that the predicted overall moduli coincide with the upper or lower-bound estimates of Hashin and Shtrikman, or it can be chosen "self-consistently". Results for particular cases are presented in a companion paper.

1. INTRODUCTION

This is the first of a short series of articles whose purpose is to explore the implications of a recently-developed variational formulation [1] for waves in randomly inhomogeneous media. Applications to composite media such as polycrystalline aggregates or fibre-reinforced materials are envisaged, so that wavelengths of disturbances are expected to be larger than typical microstructural dimensions. In fact, only the long-wavelength limit is considered explicitly, at least at present, though the formulation that will be presented could cope with shorter waves.

For the class of problems under discussion, two other approaches are available, both of which involve ensemble averaging some set of governing equations. One of these, pioneered by Karal and Keller[2] and pursued more recently by McCoy[3], involves performing a perturbation analysis and then ensemble averaging. It is effective when the material properties fluctuate by small amounts about their mean values. The other approach, which has been applied to materials which comprise a matrix throughout which is distributed a population of discrete scatterers, describes the total field as the sum of fields scattered from each inhomogeneity. The equations that result are then averaged, conditionally upon one or more scatterers being fixed, to generate a hierarchy of equations, which is closed by making some closure assumption. The closure assumption that is usually made is the quasicrystalline approximation of Lax[4]. This route has been followed, employing explicit series representations for the scattered fields, by Bose and Mal[5], Varadan *et al.*[6] and others. In this form, it is not readily applicable to other types of composite such as polycrystals. Integral equations can be generated, however, which have the potential to overcome this limitation. Mal and Knopoff[7] provided such a formulation for a matrix containing spherical inclusions and deduced from it estimates of overall moduli which are exact for a dilute suspension. More recently, Willis[8, 9] formulated integral equations for "polarizations" (which are defined in Section 2) and applied them to the study of waves in a matrix containing aligned spheroidal inclusions, using Lax's quasicrystalline approximation. Devaney[10] employed quantum mechanical formalism to obtain an equation for a "transition operator", which generates a source term corresponding to the polarizations used by Willis, and again proposed to solve the equation, using the quasicrystalline approximation, for a matrix containing inclusions. Devaney, however, generated his equation relative to a general "comparison material" which he proposed to estimate self-consistently by choosing it to have the properties that his scheme predicted for the overall response of the composite. Essentially the same scheme was developed independently for the scattering of electromagnetic waves by Tsang and Kong[11].

The present approach is rather different in character, in that it does not employ a hierarchy

of averaged equations. Instead, the starting point is the stochastic stationary principle introduced in [1], which relates to configuration-dependent trial polarization fields. It generates the hierarchy exactly but an "optimal" approximate formulation is obtained by seeking a stationary point for the variational operator, over a suitably restricted subspace of configuration-dependent fields. The equations that result are similar to those that were obtained in [8–10]. The formulation allows for a composite with any number of constituents (which need not be isotropic) and arbitrary two-point correlations, and it employs a general comparison medium; it is thus, in one respect or another, more general than those given in [8–10].

The main results of this work are explicit formulae for the effective response of the composite to low-frequency disturbances, and for lowest-order real and imaginary corrections to the low-frequency dispersion relation. Both dispersion and attenuation are thus predicted. The influence of the two-point correlations appears through simple integrals, even for a generally anisotropic composite: the attenuation term involves volume integrals of the two-point correlations while the dispersion term contains integrals over planes. Particular examples will be discussed in Paper II.

This introduction is concluded with a remark on notation. Suffixes are suppressed throughout so that quantities which, in component form would be written $L_{ijkl}e_{kl}$, $\sigma_{ij}e_{ij}$, $\sigma_{ij}n_j$, with the summation convention in operation, are written simply as Le, σe, σn: whether a quantity is a tensor, a vector or a scalar should be clear from the context in which it appears.

2. FORMULATION

For an arbitrarily inhomogeneous elastic body occupying a domain Ω, the general problem under consideration is to solve the equation of motion

$$\operatorname{div}\sigma + g = \dot{p}, x \in \Omega, t > 0, \tag{2.1}$$

where σ, p and g represent respectively stress tensor, momentum density and body force. The properties of the body appear through the constitutive relations

$$\sigma = Le, p = \rho\dot{u}, \tag{2.2}$$

where e denotes the strain tensor, u denotes particle displacement and L, ρ are, respectively, the tensor of elastic moduli (assumed symmetric) and the mass density (assumed positive). Relative to a "comparison body" with elastic moduli L_0 and density ρ_0, stress and momentum polarizations τ, π are now defined through the equations

$$\sigma = L_0 e + \tau, p = \rho_0\dot{u} + \pi \tag{2.3}$$

so that

$$\tau = (L - L_0)e, \pi = (\rho - \rho_0)\dot{u}. \tag{2.4}$$

The polarizations τ, π are, of course, not known unless the displacement field u is known. It is, nevertheless, possible to give a representation for u in terms of τ, π and the Green's function G of the comparison body: as shown in [8],

$$u = u_0 - S\tau - M\pi, \tag{2.5}$$

where S, M represent integral operators whose kernels have components

$$S_{pij}(x, x', t - t') = \frac{\partial G_{pi}}{\partial x'_j}(x, x', t - t')\Big|_{(i,j)}, \tag{2.6}$$

$$M_{pi}(x, x', t - t') = \frac{\partial G_{pi}}{\partial t}(x, x', t - t'). \tag{2.7}$$

In (2.6), the bar followed by the suffix (i, j) is meant to imply symmetrization with respect to i and j.

Boundary and initial conditions have not been mentioned explicitly but they are contained in the representation (2.5) through the requirement of selecting the appropriate homogeneous boundary conditions for G. Also, the field u_0 is defined to be the solution of the given boundary value problem, but for the comparison body rather than the actual one. Operator equations for the polarizations τ, π now follow by combining eqns (2.4) with the representation (2.5) to give

$$(L - L_0)^{-1}\tau + S_x\tau + M_x\pi = e_0, \tag{2.8}$$

$$(\rho - \rho_0)^{-1}\pi + S_t\tau + M_t\pi = \dot{u}_0, \tag{2.9}$$

where S_x, M_x are operators with kernels

$$(S_x)_{pqij} = \frac{\partial^2 G_{pi}}{\partial x_q \partial x'_j}\bigg|_{(pq),(ij)}, \tag{2.10}$$

$$(M_x)_{pqi} = \frac{\partial^2 G_{pi}}{\partial x_q \partial t}\bigg|_{(pq)}, \tag{2.11}$$

$S_t = \partial S/\partial t$ and $M_t = \partial M/\partial t$, the singularities being interpreted in the sense of generalized functions. Equations (2.8), (2.9) were derived in [8]; they were shown in [1] to be equivalent to the stationary principle

$$\delta\mathscr{H}(\tau, \pi) = 0, \tag{2.12}$$

where

$$\mathscr{H}(\tau, \pi) = \frac{1}{2}\int_\Omega \mathrm{d}x\{(2e_0 - S_x\tau - M_x\pi - (L - L_0)^{-1}\tau)^*\tau + (2\dot{u}_0 - S_t\tau - M_t\pi - (\rho - \rho_0)^{-1}\pi)^*\pi\}, \tag{2.13}$$

the symbol * denoting the operation of time-convolution.

Throughout this paper, attention will be focussed upon media which are composed of n distinct types of material, or phases, perfectly bonded together. For such media, if the rth phase has moduli L_r and density ρ_r, the tensor of moduli L and the density ρ may be represented in the forms

$$L = \sum_{r=1}^{n} L_r f_r(x), \rho = \sum_{r=1}^{n} \rho_r f_r(x), \tag{2.14}$$

where the indicator function $f_r(x)$ takes the value 1 if x is in phase r and zero otherwise. Media whose structure is random will be considered, in the sense that any particular specimen is regarded as having a label α that belongs to a sample space $\mathscr{S}$ over which a probability density $p(\alpha)$ is defined. The functions $f_r(x)$ then depend upon α and the probability $P_r(x)$ of finding phase r at x is given by

$$nP_r(x) = \langle f_r(x)\rangle = \int_{\mathscr{S}} f_r(x, \alpha)p(\alpha)\,\mathrm{d}\alpha. \tag{2.15}$$

The probability $P_{rs}(x, x')$ of finding simultaneously phases r at x and s at x' is

$$P_{rs}(x, x') = \langle f_r(x)f_s(x')\rangle. \tag{2.16}$$

Probabilities involving more points are defined similarly. The probability $P_{s|r}(x', x)$ of finding

phase s at x', conditional upon finding phase r at x is defined by the relation

$$P_{sr}(x', x) = P_{s|r}(x', x)P_r(x). \tag{2.17}$$

For such random media, a natural objective is to seek various expectation values of the solution. For example, by averaging (2.5),

$$\langle u \rangle = u_0 - S\langle \tau \rangle - M\langle \pi \rangle \tag{2.18}$$

and $\langle \tau \rangle$ can be represented in the form

$$\langle \tau \rangle = \sum_{r=1}^{n} \langle \tau \rangle_r(x, t)P_r(x) = \sum_{r=1}^{n} \langle \tau(x, t)f_r(x) \rangle. \tag{2.19}$$

In (2.19), $\langle \tau \rangle_r$ represents the expected value of τ, conditional upon finding phase r at x. Expectation values conditional upon finding specified phases at other points are defined similarly; for example,

$$\langle \tau \rangle_{r_1 r_2 r_3}(x, t; x_1, x_2, x_3)P_{r_1 r_2 r_3}(x_1, x_2, x_3) = \langle \tau(x, t)f_{r_1}(x_1)f_{r_2}(x_2)f_{r_3}(x_3) \rangle. \tag{2.20}$$

(With this notation, $\langle \tau \rangle_r$ becomes a shorthand for $\langle \tau \rangle_r$ $(x, t; x)$. Cumbersome notation seems to be unavoidable at this point; fortunately, it is not needed in the sequel).

A hierarchy of equations governing such conditional expectations of τ, π is obtained by multiplying eqns (2.8), (2.9) by the appropriate factors $f_{r_i}(x_i)$ and taking expectations. It was demonstrated by Willis[1] that the entire hierarchy is implied by the "stochastic stationary principle"

$$\delta\langle \mathscr{H}(\tau, \pi) \rangle = 0, \tag{2.21}$$

where τ, π are now regarded as functions of x, t and α. The principle (2.21) provides a method for the automatic generation of equations from which approximations to $\langle u \rangle$ may be found, which make "optimal" use of limited statistical information. For example, substituting trial fields of the form

$$\tau(x, t, \alpha) = \sum_{r=1}^{n} \tau_r(x, t)f_r(x, \alpha), \tag{2.22}$$

$$\pi(x, t, \alpha) = \sum_{r=1}^{n} \pi_r(x, t)f_r(x, \alpha) \tag{2.23}$$

and seeking a stationary point for $\langle \mathscr{H} \rangle$ with τ, π so restricted leads to the equations

$$(L_r - L_0)^{-1}\tau_r + \sum_{s=1}^{n} [S_x(\tau_s P_{s|r}) + M_x(\pi_s P_{s|r})] = e_0, \tag{2.24}$$

$$(\rho_r - \rho_0)^{-1}\pi_r + \sum_{s=1}^{n} [S_t(\tau_s P_{s|r}) + M_t(\pi_s P_{s|r})] = \dot{u}_0 \tag{2.25}$$

for the functions $\tau_r(x, t)$, $\pi_r(x, t)$. These equations were derived in[1]. They make optimal use of the two-point probabilities P_{rs} in the sense that substitution of trial fields τ, π with any dependence upon the configuration more general than that given in (2.22), (2.23) would inevitably generate equations containing probabilities involving more than two points. Equations (2.24), (2.25) are closely related to some equations studied in[8], which were derived by invoking the quasicrystalline approximation of Lax[4] to close the hierarchy. Their "optimal" status was not recognised at the time but it was noted that, in the static limit, they were derivable also from the variational principle of Hashin and Shtrikman[12–14], to which (2.12) reduces in the static limit.

3. PLANE WAVES

It is convenient, before proceeding further, to represent, in eqns (2.24), (2.25), e_0 and $\dot{u}_0$ in terms of $\langle u \rangle$, using (2.18). This gives the equations

$$(L_r - L_0)^{-1}\tau_r + \sum_{s=1}^{n} [S_x(\tau_s(P_{s|r} - P_s)) + M_x(\pi_s(P_{s|r} - P_s))] = \langle e \rangle, \tag{3.1}$$

$$(\rho_r - \rho_0)^{-1}\pi_r + \sum_{s=1}^{n} [S_t(\tau_s(P_{s|r} - P_s)) + M_t(\pi_s(P_{s|r} - P_s))] = \langle \dot{u} \rangle. \tag{3.2}$$

They are not independent of u_0, because (2.18) has also to be satisfied.

Equations (2.24), (2.25) or, equivalently, (3.1) and (3.2) with (2.18), provide an approximate description of any boundary value problem for any randomly inhomogeneous medium. The present object, however, is to gain some insight into their general character by studying possible plane-wave solutions. For this, the body is taken as infinite and statistically uniform (so that P_s, $P_{s|r}$ are insensitive to translations) and boundary conditions are discarded by setting $u_0 = 0$. The required plane-wave solutions take the form

$$\tau_r(x, t) = \tau_r \exp[-i(kn.x + \omega t)], \tag{3.3}$$

$$\pi_r(x, t) = \pi_r \exp[-i(kn.x + \omega t)] \tag{3.4}$$

where τ_r, π_r on the right sides of eqns (3.3), (3.4) are constants. The circular frequency ω and unit vector n, which defines the normal to the plane wave, are taken as given and the wave-number k is to be found. It follows now, from (2.18) with $u_0 = 0$, that

$$\langle u \rangle = -(\tilde{S}\bar{\tau} + \tilde{M}\bar{\pi}) \exp[-i(kn.x + \omega t)], \tag{3.5}$$

where

$$\bar{\tau} = \sum_{r=1}^{n} P_r\tau_r, \quad \bar{\pi} = \sum_{r=1}^{n} P_r\pi_r \tag{3.6}$$

and $\tilde{S}$, $\tilde{M}$ are the Fourier transforms of the infinite-body operators S, M, evaluated at (kn, ω). Thus, they have components

$$\tilde{S}_{pij}(kn, \omega) = ikn_j\tilde{G}_{pi}(kn, \omega)|_{(i,j)}, \tag{3.7}$$

$$\tilde{M}_{pi}(kn, \omega) = -i\omega\tilde{G}_{pi}(kn, \omega), \tag{3.8}$$

where

$$\tilde{G}(kn, \omega) = [k^2L_0(n) - \rho_0\omega^2 I]^{-1}, \tag{3.9}$$

$L_0(n)$ being the acoustic tensor for the comparison material, with components

$$(L_0(n))_{ik} = (L_0)_{ijkl}n_jn_l. \tag{3.10}$$

The factor $e^{-i\omega t}$ can be suppressed throughout if time-reduced versions of the operators on the left sides of (3.1), (3.2) are employed. These follow simply (using eqns (2.6), (2.7), (2.10), (2.11)) from the time-reduced infinite-body Green's function G, for which Willis[8] derived the expression

$$G(x) = \frac{1}{8\pi^2}\sum_{N=1}^{3}\int_{|\xi|=1} dS\,\frac{U^N(\xi)U^{NT}(\xi)}{\rho_0 c_N^2}\left[\delta(\xi.x) + \frac{i\omega}{2c_N}e^{i\omega|\xi.x|/c_N}\right] \tag{3.11}$$

where the unit vector $U^N(\xi)$ represents the polarization of a plane wave with unit normal ξ, propagating in the comparison material (that is, it is an eigenvector of $L_0(\xi)$) and c_N is the corresponding wave speed. $\delta(\ )$ is the Dirac delta-function and the integration extends over the unit sphere $|\xi| = 1$ in ξ-space.

When these substitutions are made, eqns (3.1), (3.2) reduce to a set of linear homogeneous, algebraic equations for the set of constants τ_r, π_r, $r = 1, 2, \ldots n$, in which the wavenumber k plays the role of an eigenvalue. Furthermore, translation-invariance allows the equations to be generated at one point only, say $x = 0$.

4. PERTURBATION THEORY

The algebraic equations for τ_r, π_r described above generally require numerical solution. Progress can be made by analytic methods, however, in the low-frequency limit $\omega \to 0$ (or, more strictly, $\omega a/c \ll 1$, where c is a typical wave speed $c_N(\xi)$ and a is a length scale characteristic of $P_{s|r} - P_s$). An advantage of the representation (3.11) presents itself in this limit, since it decomposes G directly into the sum of the static G (obtained by setting $\omega = 0$) and a perturbation which is of order ω. The lowest-order approximation is obtained by retaining only terms of order zero in ω in eqns (3.1), (3.2). The operators M_x, S_t, M_t are all at least of order ω, while S_x, to order zero in ω, reduces to the corresponding static operator Γ^∞, whose kernel has components

$$\Gamma^\infty_{pqij}(x) = -\frac{1}{8\pi^2}\sum_{N=1}^{3}\int_{|\xi|=1} dS \frac{\xi_q U_p^N U_i^N \xi_j}{\rho_0 c_N^2}\,\delta''(\xi.x)\Big|_{(ij)(pq)}. \tag{4.1}$$

Anticipating that k is of order ω/c, the right sides $\langle e\rangle$, $\langle \dot{u}\rangle$ of (3.1), (3.2) are kept exactly, since the Fourier transforms $\tilde{S}_x$, $\tilde{M}_x$, $\tilde{S}_t$, $\tilde{M}_t$ are homogeneous of degree zero in (k, ω). To order zero in ω, then, eqns (3.1), (3.2) yield, when $x = 0$,

$$P_r(L_r - L_0)^{-1}\tau_r + \sum_{s=1}^{n} A_{rs}\tau_s = P_r\langle e\rangle, \tag{4.2}$$

$$P_r(\rho_r - \rho_0)^{-1}\pi_r = -i\omega P_r\langle u\rangle, \tag{4.3}$$

where now

$$\langle u\rangle = -(\tilde{S}\bar{\tau} + \tilde{M}\bar{\pi}) \tag{4.4}$$

$$\langle e\rangle = -(\tilde{S}_x\bar{\tau} + \tilde{M}_x\bar{\pi}) \tag{4.5}$$

and

$$A_{rs} = \int dx'\Gamma^\infty(x')(P_{sr}(x', 0) - P_sP_r). \tag{4.6}$$

The extra factor P_r is included in (4.2), (4.3) to give the symmetry $A_{rs} = A_{sr}$; this follows because $P_{sr}(x', 0) = P_{rs}(0, x') = P_{rs}(-x', 0)$, by translation invariance, and Γ^∞ is an even function of x'. The integral that defines A_{rs} converges because $P_{sr} - P_sP_r$ either tends to zero, or at least has mean value zero, at large $|x'|$, and $\Gamma^\infty(x') = 0(|x'|^{-3})$. For general k, the integrand would also contain a factor $e^{-ik\eta.x'}$ but this is dropped because of the assumption $\omega a/c \ll 1$.

Equations of the same form as (4.2)–(4.6) were discussed in [9], in the context of the study of waves in a matrix containing a single set of aligned inclusions. As in [9], they can be solved directly. A physically appealing alternative, however, is to note that eqns (4.2) define precisely the polarizations τ_r that would be generated in the composite, if it were subjected to the uniform static mean strain $\langle e\rangle$ and the polarizations were estimated using the Hashin-Shtrikman variational principle (which, as remarked earlier, is a limiting case of (2.12)). An equation

corresponding to (4.2) was first derived in this way in [15]. Suppose, formally, that eqns (4.2) are solved to yield

$$\tau_r = S_r\langle e\rangle. \tag{4.7}$$

Then, as discussed in [15], the tensor of overall moduli $\tilde{L}$ is estimated as

$$\tilde{L} = L_0 + \sum_{r=1}^{n} P_r S_r \tag{4.8}$$

so that, from (2.3), $\langle\sigma\rangle = \tilde{L}\langle e\rangle$. In terms of $\tilde{L}$, now, eqns (4.2) imply

$$\bar{\tau} = \sum_{r=1}^{n} P_r S_r\langle e\rangle = (\tilde{L} - L_0)\langle e\rangle. \tag{4.9}$$

Equations (4.3) can be put in the similar form

$$\bar{\pi} = (\tilde{\rho} - \rho_0)\langle\ddot{u}\rangle, \tag{4.10}$$

if $\tilde{\rho}$ is defined as the mean density,

$$\tilde{\rho} = \sum_{r=1}^{n} P_r\rho_r. \tag{4.11}$$

Equations (4.9) and (4.10) show that the mean polarizations are the same as those that would be produced if a wave $\langle u\rangle$ propagated freely in a uniform medium with moduli $\tilde{L}$ and density $\tilde{\rho}$. The corresponding mathematical conclusion

$$[k^2\tilde{L}(n) - \tilde{\rho}\omega^2 I]\langle u\rangle = 0 \tag{4.12}$$

which, together with (4.3) and (4.7) defines τ_r, π_r, can indeed be obtained directly from eqns (4.2) to (4.5), using the detailed forms of the Fourier transforms $\tilde{S}_x$, $\tilde{M}_x$, $\tilde{S}_t$, $\tilde{M}_t$ which are associated with $\tilde{G}$, defined by (3.9).

A solution to higher order in ω requires study of the full eqns (3.1), (3.2), which are now given in the expanded forms

$$\begin{aligned} P_r(L_r - L_0)^{-1}\tau_r + \sum_{s=1}^{n}\int \mathrm{d}x'\Gamma^{\infty}(x')(P_{sr} - P_sP_r)\,\mathrm{e}^{-ikn.x'}\tau_s \\ = P_r\langle e\rangle - \sum_{s=1}^{n}\int \mathrm{d}x'(S_x - \Gamma^{\infty})(P_{sr} - P_sP_r)\,\mathrm{e}^{-ikn.x'}\tau_s \\ - \sum_{s=1}^{n}\int \mathrm{d}x' M_x(P_{sr} - P_sP_r)\,\mathrm{e}^{ikn.x'}\pi_s, \end{aligned} \tag{4.13}$$

$$\begin{aligned} P_r(\rho_r - \rho_0)^{-1}\pi_r = -i\omega P_r\langle u\rangle + i\omega\sum_{s=1}^{n}\int \mathrm{d}x' S(P_{sr} - P_sP_r)\,\mathrm{e}^{-ikn.x'}\tau_s \\ + i\omega\sum_{s=1}^{n}\int \mathrm{d}x' M(P_{sr} - P_sP_r)\,\mathrm{e}^{-ikn.x'}\pi_s, \end{aligned} \tag{4.14}$$

in which $x = 0$ and $\langle u\rangle$, $\langle e\rangle$ remain defined by (4.4), (4.5). Lowest-order estimates for the dispersion and attenuation of waves in the composite are obtained by extracting from eqns (4.13), (4.14), the real and imaginary perturbations of lowest order to the system (4.2), (4.3). Considering first the right side of (4.13),

$$(S_x - \Gamma^{\infty})_{pqij} = \frac{1}{8\pi^2}\sum_{N=1}^{3}\int_{|\xi|=1} \mathrm{d}S\,\frac{\xi_q U_p^N U_i^N \xi_j}{\rho_0 c_N^2}\left[\frac{\omega^2}{c_N^2}\delta(\xi.x) + \frac{i\omega^3}{2c_N^3}\,\mathrm{e}^{i\omega|\xi.x|/c_N}\right]\Bigg|_{(pq)(ij)}, \tag{4.15}$$

from (2.10) and (3.11). Lowest-order perturbations are therefore obtained by neglecting exponential factors and evaluating the integrals

$$\Lambda'_{sr}(\xi) = \int \mathrm{d}x' \delta(\xi.x')(P_{sr} - P_s P_r), \tag{4.16}$$

$$\Lambda_{sr} = \int \mathrm{d}x' (P_{sr} - P_s P_r). \tag{4.17}$$

These will exist if the composite has no long-range order; this will be assumed from this point onwards. Both of the integrals have simple physical interpretations: Λ_{sr} represents P_r times the difference between the expected volume of phase s in any large volume, subject to the restriction that phase r is at the origin, and the corresponding unrestricted expected volume, while Λ'_{sr} has a similar interpretation, in terms of areas on the plane $\xi.x' = 0$.

Next, from (2.11) and (3.11),

$$(M_x)_{pqi} = \frac{-i\omega}{8\pi^2} \sum_{N=1}^{3} \int_{|\xi|=1} \mathrm{d}S \frac{\xi_p U_q^N U_i^N}{\rho_0 c_N^2} \left[\delta'(\xi.x) - \frac{\omega^2}{2c_N^2} \operatorname{sgn}(\xi.x)\, \mathrm{e}^{i\omega|\xi.x|/c_N} \right]\Big|_{(pq)}. \tag{4.18}$$

This is an odd function; correspondingly, its lowest-order contribution to (4.13) is obtained by expanding $\mathrm{e}^{-ikn.x'}$ to first order in k. The complicated term involving $\operatorname{sgn}(\xi.x)$ makes a contribution of order ω^4 which is already smaller than those obtained from (4.15). This is neglected, therefore, and only the integral

$$\int \mathrm{d}x' \delta'(\xi.x')(P_{sr} - P_s P_r)(n.x') = (n.\nabla_\xi)\Lambda'_{sr}(\xi) \tag{4.19}$$

is needed.

The terms on the right side of (4.14) may be treated similarly. The operator $M(= -i\omega G)$ is even and so generates perturbations involving $\Lambda'_{sr}(\xi)$, Λ_{sr}, while S is odd and involves the integral (4.19).

There remains the term involving Γ^∞ on the left side of (4.13). To the order now required, it is necessary to expand the exponential $\mathrm{e}^{-ik(n.x')}$ to order k^2; the term of order k contributes nothing, since Γ^∞ is even. The extra integral that it needed is

$$\int \mathrm{d}x' \delta''(\xi.x')(P_{sr} - P_s P_r)(n.x')^2 = (n.\nabla_\xi)^2 \Lambda'_{sr}(\xi). \tag{4.20}$$

The perturbed equations may now be summarised as follows.

$$P_r(L_r - L_0)^{-1}\tau_r + \sum_{s=1}^{n} A_{rs}\tau_s = P_r\langle e\rangle + \epsilon_r, \tag{4.21}$$

$$P_r(\rho_r - \rho_0)^{-1}\pi_r = -i\omega P_r\langle u\rangle + v_r, \tag{4.22}$$

where the perturbations ϵ_r, v_r are given by

$$\epsilon_r = -\sum_{s=1}^{n} [(k^2 A_{rs}^{(kk)} + \omega^2 A_{rs}^{(\omega\omega)} + i\omega^3 D_{rs})\tau_s + \omega k B_{rs}\pi_s], \tag{4.23}$$

$$v_r = \sum_{s=1}^{n} [\omega k B'_{rs}\tau_s + (\omega^2 C_{rs} + i\omega^3 E_{rs})\pi_s] \tag{4.24}$$

and the constants in eqns (4.23), (4.24) are as listed.

$$(A_{rs}^{(kk)})_{pqij} = \frac{1}{8\pi^2}\sum_{N=1}^{3}\int_{|\xi|=1} dS\,\frac{\xi_q U_p^N U_i^N \xi_j}{\rho_0 c_N^2}(n.\nabla_\xi)^2\Lambda'_{sr}(\xi)\Big|_{(pq)(ij)}, \tag{4.25}$$

$$\left(A_{rs}^{(\omega\omega)}\right)_{pqij} = \frac{1}{8\pi^2}\sum_{N=1}^{3}\int_{|\xi|=1} dS\,\frac{\xi_p U_p^N U_k^N \xi_j}{\rho_0 c_N^4}\Lambda'_{sr}(\xi)\Big|_{(pq)(ij)}, \tag{4.26}$$

$$(B_{rs})_{pqi} = -\frac{1}{8\pi^2}\sum_{N=1}^{3}\int_{|\xi|=1} dS\,\frac{\xi_q U_p^N U_i^N}{\rho_0 c_N^2}(n.\nabla_\xi)\Lambda'_{sr}(\xi)\Big|_{(pq)}, \tag{4.27}$$

$$(B'_{rs})_{ipq} = (B_{rs})_{pqi}, \tag{4.28}$$

$$(C_{rs})_{pi} = \frac{1}{8\pi^2}\sum_{N=1}^{3}\int_{|\xi|=1} dS\,\frac{U_p^N U_i^N}{\rho_0 c_N^2}\Lambda'_{sr}(\xi), \tag{4.29}$$

$$(D_{rs})_{pqij} = \frac{1}{16\pi^2}\sum_{N=1}^{3}\int_{|\xi|=1} dS\,\frac{\xi_q U_p^N U_i^N \xi_j}{\rho_0 c_N^3}\Lambda_{sr}\Big|_{(pq)(ij)}, \tag{4.30}$$

$$(E_{rs})_{pi} = \frac{1}{16\pi^2}\sum_{N=1}^{3}\int_{|\xi|=1} dS\,\frac{U_p^N U_i^N}{\rho_0 c_N^3}\Lambda_{sr}. \tag{4.31}$$

The eigenvalues of the system (4.21), (4.22) may now be sought by treating the system as a perturbation of (4.2), (4.3). First, by inverting the terms on the left sides of (4.21), (4.22),

$$\tau_r = \sum_{s=1}^{n} T_{rs}(P_s\langle e\rangle + \epsilon_s), \tag{4.32}$$

$$\pi_r = (\rho_r - \rho_0)(-i\omega\langle u\rangle + v_r/P_r), \tag{4.33}$$

in which the inverse operator T_{rs} has the symmetry $T_{rs} = T_{sr}$, from the symmetry of A_{rs} and, for consistency with (4.7),

$$S_r = \sum_{s=1}^{n} T_{rs}P_s. \tag{4.34}$$

It follows, therefore, that

$$\bar{\tau} = (\tilde{L} - L_0)\langle e\rangle + \sum_{s=1}^{n} S_s\epsilon_s, \tag{4.35}$$

$$\bar{\pi} = i\omega(\tilde{\rho} - \rho_0)\langle u\rangle - i\omega\sum_{s=1}^{n}(\rho_s - \rho_0)v_s, \tag{4.36}$$

having used the symmetry of T_{rs}, with (4.34) and (4.9), to derive (4.35). In addition, eqn (4.4) must hold exactly: upon writing this in the explicit form

$$\{[k^2L_0(n) - \rho_0\omega^2 I]\langle u\rangle\}_i = -ik\bar{\tau}_{ij}n_j + i\omega\bar{\pi}_i \tag{4.37}$$

and substituting for $\bar{\tau}$, $\bar{\pi}$ from (4.35), (4.36), the equation

$$\{[k^2\tilde{L}(n) - \tilde{\rho}\omega^2 I]\langle u\rangle\}_i = \sum_{s=1}^{n}[-k(S_s\epsilon_s)_{ij}n_j + i\omega(\rho_s - \rho_0)(v_s)_i] \tag{4.38}$$

is obtained. If the right side of (4.38) is disregarded, the long-wavelength approximation (4.12) is recovered. Let $\langle u\rangle_N$, k_N represent an eigenvector and corresponding eigenvalue of (4.12). Then the perturbed eigenvalue, $k = k_N + \delta k_N$ say, of (4.38) can be obtained by forming the inner product of (4.38) with $\langle u\rangle_N$, to give

$$k^2 - {k_N}^2 = \sum_{s=1}^{n} [\langle e\rangle_N S_s \epsilon_s + i\omega\langle u\rangle_N(\rho_s - \rho_0)v_s]/\langle u\rangle_N \tilde{L}(n)\langle u\rangle. \tag{4.39}$$

The required lowest-order approximation to $k^2 - {k_N}^2$ now follows by replacing the still-unknown k, $\langle u\rangle$, τ_s, π_s by their unperturbed values k_N, $\langle u\rangle_N$, together with the associated τ_s, π_s, on the right side of (4.39). Thus, finally, using (4.7) and (4.3) to express the right side just in terms of τ_s, π_s,

$$\frac{k^2}{{k_N}^2} - 1 = Q' + iQ \tag{4.40}$$

where

$$Q' = \frac{-1}{k^2\langle u\rangle \tilde{L}(n)\langle u\rangle} \sum_{r=1}^{n}\sum_{s=1}^{n} [\tau_r(k^2 A_{rs}^{(kk)} + \omega^2 A_{rs}^{(\omega\omega)})\tau_s + 2\omega k \tau_r B_{rs}\pi_s + \omega^2 \pi_r C_{rs}\pi_s] \tag{4.41}$$

and

$$Q = \frac{-\omega^3}{k^2\langle u\rangle \tilde{L}(n)\langle u\rangle} \sum_{r=1}^{n}\sum_{s=1}^{n} [\tau_r D_{rs}\tau_s + \pi_r E_{rs}\pi_s]. \tag{4.42}$$

In (4.41), (4.42) the suffix N has been suppressed for notational convenience: it is to be understood that k, $\langle u\rangle$ take the values k_N, $\langle u\rangle_N$ and that τ_r, π_r are the associated polarizations, related to them by (4.3) and (4.7). Both τ_r and π_r are of order $\omega\langle u\rangle$. Correspondingly, Q' and Q are of order ω^2, ω^3 respectively. Since the wave is characterized by the factor $\exp[-i(kn.x + \omega t)]$, positive Q corresponds to attenuation in the direction of propagation while positive Q' corresponds to a reduction (of order ω^2) in the phase velocity, $\mathrm{Re}(\omega/k)$.

5. DISCUSSION

The preceding sections have summarised a set of equations (namely, (2.24) and (2.25)) that provide an approximate description for waves in a composite, making "optimal" use of two-point correlations, their optimal status being judged relative to the stationary principle (2.21). Possible plane-wave solutions have been discussed and dispersion relations have been given explicitly, through equations (4.12) and (4.40), for long waves. No assumptions have been made concerning the elastic symmetry of the phases, symmetries of the correlations or the type of composite: the formulae embrace equally well cases such as a matrix containing a distribution of discrete inclusions and a polycrystalline aggregate. Examples will be given in the paper that follows (Paper II). The present article will be concluded with a number of general remarks.

It may be noted first that the formulation employs a general comparison material, with moduli L_0 and density ρ_0. Once these are chosen, the formulation is "optimal", but different choices of L_0, ρ_0 of course yield different "optimal" estimates for τ_r, π_r and the dispersion relation $k(\omega)$. In the long-wavelength limit, the formulation generates the Hashin–Shtrikman estimates $\tilde{L}$ for the overall moduli and the true overall moduli, $\bar{L}$ say, are bounded in the sense that $\bar{L} - \tilde{L}$ is positive- or negative-definite whenever $L_r - L_0$ is correspondingly definite for each r. This result was generalized in [1], where it was shown that the stationary principle (2.12) in fact provides a minimum principle for the Laplace transform of $\mathcal{H}$ when the transform variable is real and positive, so long as L_0, ρ_0 are chosen so that $L_r - L_0$ is negative-definite and $\rho_r - \rho_0$ is

negative for each r, and a maximum principle if $L_r - L_0$ is positive-definite and $\rho_r - \rho_0$ is positive for each r.

Just as the Hashin–Shtrikman upper- and lower-bound estimates for $\tilde{L}$ are not always close together, it is to be expected that the dispersion relation $k(\omega)$ predicted from the present equations is sensitive to the choice of L_0, ρ_0 and the problem of selecting the best L_0, ρ_0 presents itself. In the static limit, Willis[15] showed that the self-consistent estimates of Hershey[17], Budiansky[18], Hill[19] and others were generated by stipulating that L_0 should be chosen so that the mean polarization $\bar{\tau}$ was zero which, in turn, implies that $\tilde{L} = L_0$. This prescription is not, of course, exact, because the assumed piecewise-constant form for τ (to which (2.22) reduces) is not likely to provide the exact solution; it is just the best that can be done if only two-point correlations are known. In Paper II which follows, estimates for $\tilde{L}$ and for the perturbations Q, Q' in (4.40) are calculated for particular composites, with L_0, ρ_0 chosen both to provide Hashin–Shtrikman upper- and lower-bound estimates, and chosen "self-consistently", so that $L_0 = \tilde{L}$, $\rho_0 = \tilde{\rho}$.

A more general self-consistent scheme for dynamic problems has recently been proposed by Devaney[10]. He employed methods developed in quantum mechanics by Gyorffy[20] and Korringa and Mills[21] and used quantum-mechanical formalism. In essence, however, the idea was to employ a comparison material characterised by non-local operators L_0, ρ_0 (whose Fourier transforms are functions of k and ω) and to choose L_0, ρ_0 so that a certain transition operator had mean value zero. The transition operator was to be estimated by employing the quasicrystalline approximation of Lax[4]. In present notation, the transition operator applied to u_0 generates the source term $\operatorname{div} \tau - \dot{\pi}$; the procedure is thus equivalent to applying the quasi-crystalline approximation to τ, π and then choosing L_0, ρ_0 so that $\langle \tau \rangle = \langle \pi \rangle = 0$. Equations corresponding to (2.24), (2.25) were developed in[8] for a nonlocal, viscoelastic comparison material, with this possibility in mind and, although this was not mentioned in[1], the reasoning that leads to the stationary principle (2.21) applies unaltered if L_0 and ρ_0 are such operators, so long as they satisfy certain plausible symmetry conditions. With this slight restriction, eqns (2.24), (2.25) could be used as a starting point for Devaney's scheme, with the advantage that they have an "optimal" status, independent of the quasicrystalline assumption. Devaney actually considered only the case of a matrix containing inclusions, distributed so that the composite overall was isotropic. To this extent, eqns (2.24), (2.25) provide a generalization. It should, perhaps, be mentioned that Devaney's scheme was implemented in[10] only in the Born approximation, which applies to weak scatterers. The results contained in Section 4 are the only ones known to us that are applicable to composites with any phase geometry, with the single restriction that there should be no long-range order.

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